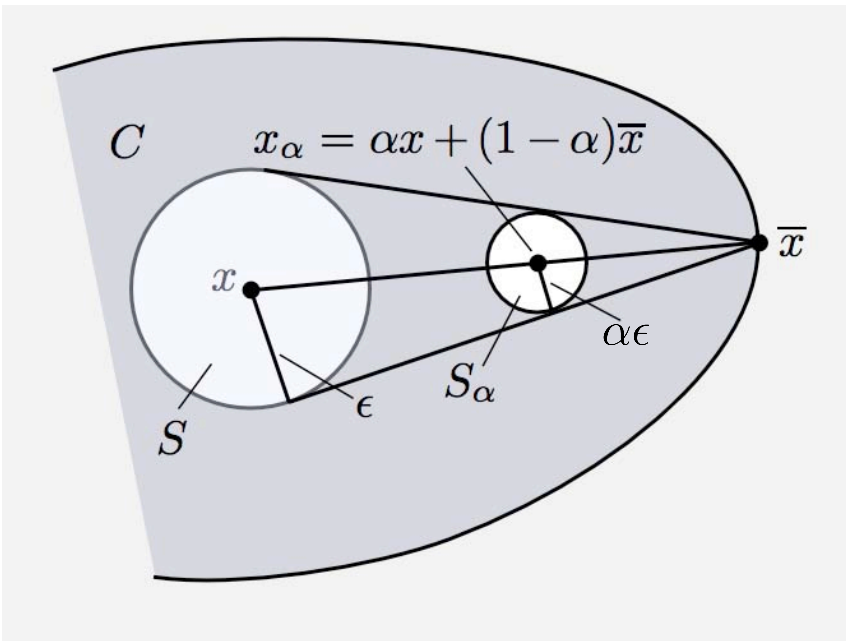
RELATIVE INTERIOR

- x is a *relative interior point* of C, if x is an interior point of C relative to aff(C).

- ri(C) denotes the *relative interior of* C, i.e., the set of all relative interior points of C.

- **Line Segment Principle**: If C is a convex set, $x \in \text{ri}(C)$ and $\overline{x} \in \text{cl}(C)$, then all points on the line segment connecting x and $\overline{x}$, except possibly $\overline{x}$, belong to ri(C).



- Proof of case where $\overline{x} \in C$: See the figure.

- Proof of case where $\overline{x} \notin C$: Take sequence $\{x_k\} \subset C$ with $x_k \to \overline{x}$. Argue as in the figure.

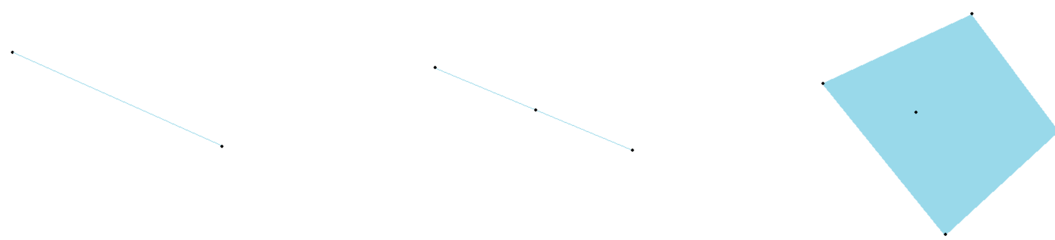


Figure 4: Convex hulls of sets of points

4 Convex sets

Convex sets are defined via affine combinations of two elements with nonnegative coefficients.

Definition 4.1. A subset $X \subset A$ of a real vector space or a real affine space is called *convex* if for all $x, y \in X$ and all $\lambda \in [0,1]$ we have

$$\lambda x + (1-\lambda)y \in X.$$

Examples:

- the empty set $\emptyset$,
- the whole space A,
- singletons $\{x\}$,
- affine subspaces,
- open or closed affine half-spaces,
- open or closed norm balls $x + rB_1^o$, $x + rB_1$ around arbitrary points.

Here open and closed affine half-spaces are sets of the form $\{x \in A \,|\, a(x) < b\}$ and $\{x \in A \,|\, a(x) \leq b\}$, respectively, where a is a non-constant linear functional on A and $b \in \mathbb{R}$.

4.1 Convex hull

Definition 4.2. Let $x_1, \dots, x_k$ be points in an affine space A. Then $\sum_{i=1}^k \lambda_i x_i$ is called a *convex combination* of the points $x_1, \dots, x_k$ if $\sum_{i=1}^k \lambda_i = 1$ and $\lambda_i \geq 0$, $i = 1, \dots, k$.

The *convex hull* of a subset $X \subset A$ of an affine space is the set of all convex combinations of elements of X. It is denoted by $convX$.

Lemma 4.3. *A set X is convex if and only if it equals its convex hull.*

Proof. Let $X = convX$. Then, in particular, convex combinations of any two elements of X belong to X. Hence X is convex.

Let X be convex. We show by induction on k that a convex combination of k elements of X is in X. The definition of convexity yields the base of the induction for $k = 2$. Suppose we have proven that any convex combination of $k-1$ elements of X is in X. Let $x_1, \dots, x_k \in X$ and let $x = \sum_{i=1}^k \lambda_i x_i$ be a convex combination. If any of the coefficients λ_i vanishes, then x is actually a convex combination of strictly less than k elements and is in X by the induction hypothesis. Assume $\lambda_i > 0$ for all $i = 1, \dots, k$. Then we have

$$x = \sum_{i=1}^{k-1} \lambda_i x_i + \lambda_k x_k = \left(\sum_{i=1}^{k-1} \lambda_i \right) \sum_{i=1}^{k-1} \frac{\lambda_i}{\sum_{j=1}^{k-1} \lambda_j} x_i + \lambda_k x_k = (1-\lambda_k)y + \lambda_k x_k.$$

Here $y = \sum_{i=1}^{k-1} \frac{\lambda_i}{\sum_{j=1}^{k-1} \lambda_j} x_i$ is a convex combination of $k-1$ elements of X and is hence in X. The point x has then been represented as convex combination of two elements of X and is hence also in X. □

The following assertion follows immediately from Definition 4.1.

Lemma 4.4. *Arbitrary intersections of convex sets are convex.*

Corollary 4.5. *The convex hull of a set X is the smallest convex set which contains X, namely the intersection of all convex sets containing X.*

Proof. Since convex combinations of convex combinations are again convex combinations of the original points, the convex hull of X is equal to its own convex hull. By Lemma 4.3 it is hence convex. On the other hand, any convex set Y containing X must contain at least the convex hull of X, because $Y \supset X$ implies $Y = conv Y \supset conv X$. □

Further examples of convex sets:

- polytopes (convex hulls of a finite set of points),
- polyhedra (finite intersections of closed affine half-spaces),
- simplices (convex hull of an affinely independent set of points).

4.2 Operations preserving convexity

We now consider more operations which preserve convexity.

Definition 4.6. Let X, Y be subsets of a vector space. The set

$$X + Y := \{x + y \,|\, x \in X,\ y \in Y\}$$

is called *Minkowski sum* of X, Y.

This definition can be extended to the case where one of the sets X, Y is a subset of an affine space and the other a subset of the underlying vector space.

The following assertions follow easily from the definition of convexity.

- the Minkowski sum of convex sets is convex,
- images of convex sets under affine maps are convex,
- pre-images of convex sets under affine maps are convex,
- the interior X^o of a convex set X is convex,
- the relative interior $ri\, X$ of a convex set X is convex,
- the closure $cl\, X$ of a convex set X is convex.

We now come to the interplay between convexity and topology.

Lemma 4.7. *Let $X \neq \emptyset$ be convex. Then $ri\, X \neq \emptyset$.*

For non-convex sets this is in general not the case (consider $X = \mathbb{Q} \subset \mathbb{R}$, then $ri\, X = \emptyset$).

Proof. The affine hull $aff\, X$ possesses an affine basis of points in X. To construct such a basis, pick an arbitrary point $x_1 \in X$. If $aff\,\{x_1\} = aff\, X$, then $\{x_1\}$ is an affine basis of $aff\, X$. If $aff\,\{x_1\} \neq aff\, X$, then there exists a point $x_2 \in X \setminus aff\,\{x_1\}$. This point x_2 is affinely independent of x_1. We now repeat the process by comparing $aff\,\{x_1, x_2\}$ with $aff\, X$ and adjoin another affinely independent point $x_3 \in X$ if these affine hulls are not equal. Obviously the affine hulls become equal after $\dim aff\, X + 1$ steps.

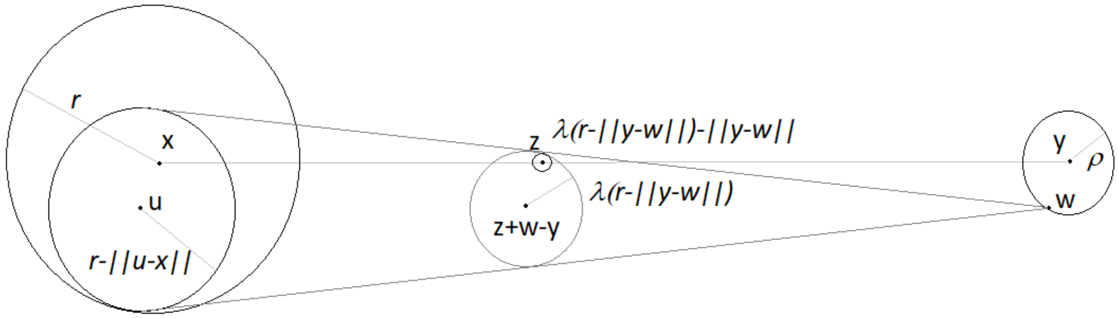


Figure 5: Proof of Lemma 4.8. Radii are shown in *italic*.

Let hence $x_1, \dots, x_k \in X$ form an affine basis of the affine hull of X. Then the simplex $\Sigma = conv\{x_1, \dots, x_k\}$ is a subset of X, and the relative interior of Σ is given by the set

$$ri\,\Sigma = \left\{ \sum_{i=1}^{k} \lambda_i x_i \,|\, \lambda_i > 0, \ \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

Since $aff\,\Sigma = aff\,X$, any point in $ri\,\Sigma$ is also in $ri\,X$. □

We now need an auxiliary lemma.

Lemma 4.8. *Let X be a convex set, let $x \in ri\,X$ and $y \in clX$. Then the half-open segment $[x, y) = \{\lambda x + (1 - \lambda)y \,|\, \lambda \in (0, 1]\}$ is a subset of $ri\,X$.*

Proof. By definition there exists $r > 0$ such that $(x + rB_1) \cap aff\,X \subset X$. Let $\lambda \in (0, 1]$ and $z = \lambda x + (1 - \lambda)y$. Set $\rho = \frac{\lambda r}{1+\lambda}$. Since $y \in clX$, there exists $w \in X$ such that $||y - w|| < \rho$.

Set $u = x+w-y$. Then $u \in aff\,X$ as an affine combination of points in $aff\,X$. Moreover, $||u-x|| = ||w-y|| < r$. Hence $(u + (r - ||u - x||)B_1) \cap aff\,X \subset (x + rB_1) \cap aff\,X \subset X$. We then get

$$\lambda[(u + (r - ||u - x||)B_1) \cap aff\,X] + (1 - \lambda)w = [z + w - y + \lambda(r - ||y - w||)B_1] \cap aff\,X \subset X$$

by the convexity of X. But

$$z + w - y + \lambda(r - ||y - w||)B_1 \supset z + (\lambda(r - ||y - w||) - ||y - w||)B_1$$

and $\lambda(r - ||y - w||) - ||y - w|| = (1 + \lambda)(\rho - ||y - w||) > 0$. Therefore $(z + (1 + \lambda)(\rho - ||y - w||)B_1) \cap aff\,X \subset X$, and $z \in ri\,X$. □

This will allow us to show that for convex sets the relative interior and the closure can be obtained from each other.

Lemma 4.9. *Let X be a convex set. Then $cl\,ri\,X = clX$ and $ri\,clX = ri\,X$.*

Proof. Clearly $cl\,ri\,X \subset clX$ and $ri\,clX \supset ri\,X$.

Let $y \in clX$. Then $X \neq \emptyset$ and there exists a point $x \in ri\,X$. It follows that $[x, y) \subset ri\,X$, and hence $y \in cl\,ri\,X$.

Let now $z \in ri\,clX$. Then $X \neq \emptyset$ and there exists $x \in ri\,X$. Further there exists $\varepsilon > 0$ such that $(z + \varepsilon B_1) \cap aff\,X \subset clX$. We have $[x, z] \subset aff\,X$, and there exists $y \in (z + \varepsilon B_1) \cap aff\,X$ such that y lies on the line through x and z and such that $z \in [x, y)$. But then $z \in ri\,X$ by Lemma 4.8. □